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# Decoupling Core Velocity and Magnetic Induction in Earth's Dynamo

***As specified under the MHD approximation for Earth's dynamo, Ampere's law in combination with Ohm's law is inverted, without additional approximations or assumptions, to yield an analytic expression for the Earth's core-fluid velocity solely in terms of the magnetic induction. This velocity can be loosely characterized as being derived from forces associated with magnetic pressure and stress or tension along the magnetic field lines. Inserting this analytic expression for the fluid velocity into the usual magnetic induction equation, derived under the MHD approximation appropriate for Earth's dynamo, effectively decouples the magnetic induction from the fluid velocity, thereby yielding an inhomogeneous magnetic-diffusion equation involving only the magnetic induction. The source term in this decoupled equation has a self-regulating factor proportional to the inverse square of the total intensity. So, as the total intensity becomes large, the source term becomes small and the homogeneous diffusion equation is approached, whereby, the field decays. As the total intensity decreases, the source term then increases and the induction field grows. This self modulating effect, therefore, tends to maintain the field magnitude within some bounds while the polarity is free to reverse or not as other source-term elements dictate. The main consequence of this decoupling is that further analysis of the dynamo problem can proceed as if dealing with a kinematic dynamo, with the important distinction that the fluid velocity is not arbitrarily specified, but follows directly from Maxwell's equations and Ohm's law. For this reason, the resulting class of dynamo is highly restrictive, but potentially fundamental to the geodynamo problem.***

## Introduction

Maxwell's equations and Ohm's law as specified for the geodynamo problem, can, in a well known manner (e.g., Merrill and McElhinny [1983]) employing the usual Magneto-Hydro-Dynamic (MHD) approximation, be combined in such a way as to eliminate the electric field to yield the familiar magnetic induction equation. This equation may be characterized as an inhomogeneous magnetic diffusion equation with a source term involving the cross product $\mathbf{v}\times\mathbf{B}$, where v is the fluid velocity in the core and **B** is the magnetic induction field. Formally decoupling these parameters in the induction equation without further assumptions or approximations has hitherto not been realized.

Kinematic dynamo theory, as discussed for instance by Roberts and Gubbins (1987), decouples the fluid velocity from the magnetic induction by specifying a priori, an arbitrary velocity field, which then allows the induction equation to be solved for the magnetic induction field given some initial conditions and boundary conditions. The object of this approach is to determine if the specified velocity field can sustain dynamo action and thereby exhibit magnetic reversals. However, even if dynamo action is sustained, there is no reason to suppose that an arbitrarily chosen velocity field is in any way representative of the true velocity field in the Earth's core. Some additional constraints are needed.

A reexamination of Maxwell's equations for the geodynamo leads to a new constraint which eliminates the need to choose the velocity field arbitrarily. This constraint is in essence an analytic expression for the velocity field which depends exclusively on the magnetic induction field. It consists of a complicated set of terms loosely corresponding to velocities generated by magnetic pressure and stress or tension along the magnetic field lines. Inserting this velocity into the magnetic induction equation yields a nonlinear inhomogeneous magnetic-diffusion equation that is dependent only on the magnetic induction field. The magnetic induction equation is thus, without arbitrary assumptions, formally decoupled from the fluid velocity. Once the magnetic induction field is determined from the diffusion equation, the velocity field, the electric field, and the electric current in the core are determined as well.

## The Decoupling Procedure

Under the usual MHD approximation, Maxwell's electromagnetic field equations for the geodynamo problem expressed in Gaussian units are as follows:

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} \tag{1a}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \tag{1b}$$

$$\nabla \bullet \boldsymbol{E} = 0 \tag{1c}$$

$$\nabla \bullet \boldsymbol{B} = 0 \tag{1d}$$

where the current density **J** in terms of the core-fluid's conductivity **s** and velocity **v** is given by Ohm's law as:

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \tag{2}$$

The magnetic induction equation for the dynamo is generated by taking the curl of eq. (1a) and using eqs. (1b), (1d), and (2). As an aside, a similar equation for the electric field can be generated by taking the curl of eq. (1b) and using eqs. (1a), (1c), and (2). The following familiar induction equations result:

$$\eta\nabla^2\boldsymbol{B} - \frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{3a}$$

$$\eta\nabla^2\boldsymbol{E} - \frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{v} \times \boldsymbol{B}) \tag{3b}$$

where the magnetic diffusivity η is defined as:

$$\eta = \frac{c^2}{4\pi\sigma} \tag{4}$$

Now combining eq. (1a), Ampere's law, with eq. (2), Ohm's law, and recasting the result into tensor component form yields:

$$\varepsilon^{ijk}B_{j/i} = \frac{1}{\eta}(cE^k + \varepsilon^{ijk}v_iB_j) \tag{5}$$

where the Latin indices i, j, and k range from 1 to 3 corresponding to the three coordinates x, y, and z respectively; where Einstein summation notation is assumed, whereby repeated pairs of indices (one raised and one lowered) are to be summed over; and where the Levi-Civita symbol has been introduced, which yields: the value of 1 if its three indices are even cyclic permutations of the numbers 1, 2, and 3 (e.g., 312); the value of -1 if its three indices are odd permutations of these numbers (e.g., 213); and the value of zero if any two indices are equal. The slash symbol (/) denotes partial differentiation with respect to the indicated coordinate index (e.g., $\partial B_x/\partial z \equiv B_{1/3}$).

The components of the velocity **v** can be isolated by first noting the Levi-Civita identity:

$$\varepsilon^{ijk}\varepsilon_{ijl} = 2\delta^k_l \tag{6}$$

so that:

$$E^k = \delta^k_l E^l = \frac{1}{2}\varepsilon^{ijk}\varepsilon_{ijl}E^l \tag{7}$$

Inserting eq. (7) into eq.(5) and rearranging terms slightly, yields the following result:

$$\varepsilon^{ijk}[B_{j/i} - \frac{1}{\eta}(\frac{c}{2}\varepsilon_{ijl}E^l + v_i B_j)] = 0 \qquad (8)$$

Clearly, eq. (8) is satisfied for each value of k, if:

$$B_{j/i} = \frac{1}{\eta}(\frac{c}{2}\varepsilon_{ijl}E^l + v_i B_j) \qquad (9)$$

There may be other ways of satisfying eq. (8). So, eq. (9) is not necessarily unique, although it seems to be the most natural solution, and one that has interesting consequences.

Multiplying through by $\mathbf{B}^j$ and noting that:

$$B^2 = B_j B^j \qquad (10a)$$

$$(B^2)_{/i} = 2B^j B_{j/i} \qquad (10b)$$

eq. (9) can be solved for the velocity components, yielding:

$$v_i = \frac{1}{2B^2}[\eta(B^2)_{/i} - c\varepsilon_{ijl}B^j E^l] \qquad (11)$$

Reverting back to vector notation, this equation can be written as:

$$\boldsymbol{v} = \frac{1}{2B^2}[\eta\nabla(B^2) + c\boldsymbol{E}\times\boldsymbol{B}] \qquad (12)$$

The first term on the right-hand side can be identified as a velocity induced by magnetic pressure. The second term is an $\mathbf{E}\times\mathbf{B}$ drift velocity.

Now, eq. (5) can be written in vector form as:

$$\boldsymbol{E} = \frac{1}{c}(\eta\nabla\times\boldsymbol{B} - \boldsymbol{v}\times\boldsymbol{B}) \qquad (13)$$

Inserting eq. (13) into eq. (12) then yields the following transcendental velocity equation:

$$\boldsymbol{v} = \frac{\eta}{2B^2}[\nabla(B^2) + (\nabla\times\boldsymbol{B})\times\boldsymbol{B}] - \frac{1}{2B^2}(\boldsymbol{v}\times\boldsymbol{B})\times\boldsymbol{B} \qquad (14)$$

This is a transcendental equation for the fluid velocity solely in terms of the magnetic induction field. By inserting the following vector identities into eq. (14):

$$(\boldsymbol{v}\times\boldsymbol{B})\times\boldsymbol{B} = (\boldsymbol{B}\bullet\boldsymbol{v})\boldsymbol{B} - B^2\boldsymbol{v} \qquad (15a)$$

$$(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = (\boldsymbol{B}\bullet\nabla)\boldsymbol{B} - \frac{1}{2}\nabla(B^2) \qquad (15b)$$

an alternative form of the transcendental velocity equation results:

$$\boldsymbol{v} = \frac{\eta}{B^2}[\frac{1}{2}\nabla(B^2) + (\boldsymbol{B}\bullet\nabla)\boldsymbol{B}] - \frac{1}{B^2}(\boldsymbol{B}\bullet\boldsymbol{v})\boldsymbol{B} \qquad (16)$$

By taking the vector dot product of this equation with the magnetic induction field **B**, solving for the dot product (**B** • **v**), and inserting the result into the last term in eq. (16), the fluid velocity can be expressed entirely in terms of the magnetic induction field, as follows:

$$\boldsymbol{v} = \frac{\eta}{B^2}[\frac{1}{2}\nabla(B^2) + (\boldsymbol{B}\bullet\nabla)\boldsymbol{B}] - \frac{\eta}{2B^4}\{[\frac{1}{2}\nabla(B^2) + (\boldsymbol{B}\bullet\nabla)\boldsymbol{B}]\bullet\boldsymbol{B}\}\boldsymbol{B} \qquad (17)$$

Inserting this result into eq. (13) and noting that $\mathbf{B}\times\mathbf{B}$ is zero, so that the entire last term of eq. (17) does not contribute, the electric field becomes

$$\boldsymbol{E} = \frac{\eta}{c}\{\nabla\times\boldsymbol{B} - \frac{1}{B^2}[\frac{1}{2}\nabla(B^2) + (\boldsymbol{B}\bullet\nabla)\boldsymbol{B}]\times\boldsymbol{B}\} \qquad (18)$$

Similarly, inserting eq. (17) into eq. (3a), the magnetic induction equation, the following inhomogeneous magnetic-diffusion equation, which depends solely on the magnetic induction field **B**, results:

$$\frac{1}{\eta}\frac{\partial\boldsymbol{B}}{\partial t} - \nabla^2\boldsymbol{B} = \nabla\times\{\frac{1}{B^2}[\frac{1}{2}\nabla(B^2) + (\boldsymbol{B}\bullet\nabla)\boldsymbol{B}]\times\boldsymbol{B}\} \qquad (19)$$

Equations (17), (18), and (19) are the main results. It is clear that there has been a complete decoupling of the fluid velocity from the magnetic induction field in the magnetic induction equation. This is a pleasant simplification. However, eq. (19) is still formidable. The prescription is to solve eq. (19) for the magnetic induction field and insert the result into eqs. (17) and (18) to obtain the fluid velocity and electric field in Earth's core. Evaluating eqs. (17) and (18) given the magnetic induction field from eq. (19), though straight forward, is nevertheless nontrivial. The source term in eq. (19) appears to be a complicated function of magnetic pressure and magnetic stress or tension along the field lines. Note further that the divergence of the velocity in eq. (17) is not zero in general. So, the fluid incompressibility condition often imposed in dynamo theory is not satisfied.

## Conclusion

Other than the usual MHD approximation, no additional approximations were used in the derivation of the decoupled magnetic-induction (diffusion) equation. Intuitively this equation is expected to exhibit dynamo action. Since the only means of altering this equation is through the magnetic diffusivity, it stands as a highly restrictive dynamo equation which is potentially fundamental to the dynamo problem. In support of the view that eq. (19) will lead to a viable dynamo, it can be noted that the velocity in eq. (17) is invariant with respect to a change in the sign of the magnetic induction. Likewise, eq. (19) is also invariant with respect to such a change in sign. The electric field reverses in response to magnetic field reversals. So, both **B** and **-B** are acceptable solutions. Further-more, note that the source term on the right-hand side of eq. (19) has a factor that is inversely proportional to the square of the total intensity. This factor appears to serve as a self-regulating mechanism for the dynamo. That is, as the field intensity grows, the source term tends to zero yielding a homogeneous diffusion equation, for which the field will decay. Then, as the field intensity becomes small, the source term increases, causing the field to grow. Thereby, the magnetic field intensity is constrained to remain within some bounds. Meanwhile, the polarity of the magnetic induction field is free to reverse or not, depending on the value of other elements in the source term, and their derivatives. Thus, in view of the complicated nonlinear nature of the source term in the magnetic diffusion equation, which seems sufficiently robust to admit reversals, the diffusion equation appears to posses the necessary ingredients to generate self-exciting dynamo action. However, this proposition still needs computational verification.

## References

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Submitted to *Journal of Earth, Planets, and Space*